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Numerical representation of *PQI* interval orders

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Abstract

We consider the problem of numerical representations of *PQI* interval orders. A preference structure on a finite set A with three relations P, Q, I standing for "strict preference", "weak preference" and "indifference", respectively, is defined as a *PQI* interval order iff there exists a representation of each element of A by an interval in such a way that, P holds when one interval is completely to the right of the other, I holds when one interval is included to the other and Q holds when one interval is to the right of the other, but they do have a non-empty intersection (Q modelling the hesitation between P and I). Only recently, necessary and sufficient conditions for a *PQI* preference structure to be identified as a *PQI* interval order have been established. In this paper, we are interested in the problem of constructing a numerical representation of a *PQI* interval order and possibly a minimal one. We present two algorithms, the first one in $O(n^2)$ aimed to determine a general numerical representation, and the second one, in $O(n)$, aimed to minimise such a representation.

Keywords: Intervals; PQI interval orders; Numerical representation; Minimal representation

1. Introduction

In preference modelling and decision support we often have to compare intervals instead of discrete values. This is due to the fact that the comparison of alternatives is usually realised through their evaluation on numerical scales, subject to the unavoidable lack of precision and

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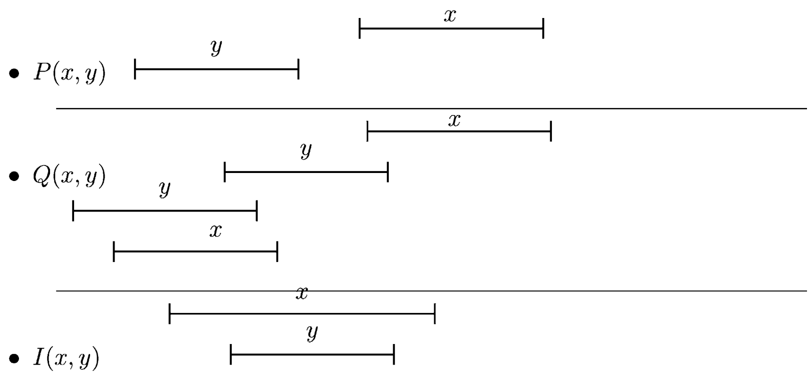

Fig. 1. Relations *P*, *Q* and *I*.

certainty. The conventional structure adopted in order to compare two intervals, considers that "*x* is preferred to *y*" ($P(x, y)$) iff the interval associated to *x* is completely to the "right" (in the sense of the line representing the reals) of the interval associated to *y*. In all other cases "*x* is indifferent to *y*". Such a model (where indifference is not transitive) may conceal the fact that "*x* being to the right of *y*" (the intersection being not empty) is a situation intuitively different from the case where one interval (let us say *x*) is included in the other (let us say *y*). The second case can be considered a "sure indifference" as much as can be considered a "sure preference" the case $P(x, y)$. Under such a perspective the first case is a situation of hesitation between preference and indifference, which merits to be considered separately (see [11]). We may denote such a situation as "weak preference" and represent it as $Q(x, y)$. We come up with a preference structure known as *PQI* interval order (*PQI*–IO). For an intuitive representation of this concept see Fig. 1.

The *PQI*–IO has been discussed since 1988 by Vincke [14]. The problem of characterising such a structure was left open until recently. Tsoukiàs and Vincke [12,13] provided necessary and sufficient conditions for a *PQI* preference structure to be identified as a *PQI*–IO. The operational problem of detecting if a given *PQI* preference structure satisfies such conditions was solved in Ngo The et al. [5], through an algorithm which is demonstrated to run in polynomial time.

In this paper, we are interested in the problem of numerical representations of a *PQI*–IO. For this purpose, our paper is dedicated to investigate some aspects of the representation of a *PQI*–IO (once detected). First we show the importance of considering what we call a "separated *PQI*–IO" (where indifference is separated in two partial orders, one the inverse of the other). Then we exploit well-known results concerning conventional interval orders and extend them to the case of *PQI*–IO. Practically, we obtain a result enabling to order the endpoints of the intervals of a *PQI*–IO. These theoretical results lead to two algorithms: the first one determines a general numerical representation and the second one a minimal one. On the notion of minimal representation the reader can see Pirlot and Vincke [7, Chapter 4].

Our findings extend (partially) results obtained in the frame of the "Interval Satisfiability" (ISAT) problem (see [4,6]). In this case the question is to find a realisation (a numerical representation) for a set of "events" possibly temporal ones, see also [1] when a number of possible relations hold among them. This is a concept similar to ours. However, in the ISAT case only intersection and not intersection (possibly oriented) are distinguished, while in our work we distinguish oriented intersection from oriented inclusion. On the other hand our work considers that one and only one relation holds for a given pair of "events", while in the ISAT case several possibilities are allowed.

The paper is organised as follows. Section 2 provides the basic notations and definitions. In Section 3 we recall some definitions and previous results concerning the numerical representation of interval orders. Section 4 is dedicated to *PQI*–IO. Section 5 gives the two algorithms constructing a general representation of a *PQI* interval order and a minimal one. Appendix A contains the (long) proofs of some theorems and propositions within the paper.

2. Basic notations, definitions and known results

In the following, if not indicated differently, all the relations under consideration are binary relations defined on a finite set A and denoted by P, Q, I, R, S, T. The fact that $(x, y) \in S$ is denoted either by $S(x, y)$ or xSy. We adopt the following notation (cf. [8]).

$$S^{-1} = \{(x, y) : S(y, x)\}, \quad S^c = \neg S = \{(x, y) : \neg S(x, y)\},$$

$$S^d = \neg S^{-1} = \{(x, y) : \neg S(y, x)\}, \quad S^\sim = A^2 \backslash (S \cup S^{-1}),$$

$$S \subset T : \forall x, y, S(x, y) \Rightarrow T(x, y), \quad S^+(a) = \{x \in A : S(a, x)\},$$

$$S \cup T = \{(x, y) : S(x, y) \vee T(x, y)\}, \quad S \cap T = \{(x, y) : S(x, y) \wedge T(x, y)\},$$

$$S^\approx = \{(x, y) : \forall z, S(x, z) \Leftrightarrow S(y, z) \quad \text{and} \quad S(z, x) \Leftrightarrow S(z, y)\},$$

$$S.T = \{(x, y) : \exists z, S(x, z) \wedge T(z, y)\}, S^2 = S.S.$$

If S is an equivalence relation on A then the equivalence class containing $a \in A$ is denoted by $[a]_S$. When there is no ambiguity, we can use simply $[a]$. A binary relation S on a finite set $A = \{a_1, a_2, \dots, a_n\}$ can be represented by an $n \times n$, $0-1$ matrix M^S with $M^S_{ij} = 1$ iff $(a_i, a_j) \in R$. Further on we use the following definitions (see [8]).

Definition 2.1. A binary relation S is:

- a partial order iff it is asymmetric and transitive;
- a weak order iff it is asymmetric and negatively transitive;
- a linear order iff it is irreflexive, complete and transitive;
- an equivalence relation iff it is reflexive, symmetric and transitive.

We have the two following fundamental results from Fishburn [2]:

Theorem 2.1. *If S is a partial order then*

(i) $S^\approx$ *is an equivalence relation*;

(ii) $S = S.S^{\approx} = S^{\approx}.S$;
(iii) $S^{\approx}(x, y) \Leftrightarrow \{z : S(x, z)\} = \{z : S(y, z)\}$ *and* $\{z : S(z, x)\} = \{z : S(z, y)\}$.
(iv) $(A/S^{\approx}, S)$ *is a partial order*;

Theorem 2.2. *If S is a partial order then the following are equivalent*:

(i) *S is a weak order*;
(ii) $S^{\sim}$ *is transitive*;
(iii) $S^{\sim} = S^{\approx}$;
(iv) $S = S.S^{\sim} = S^{\sim}.S$;
(v) $(A/S^{\sim}, S)$ *is a linear order*.
In addition, S is a linear order iff $S^{\sim}$ *is the identity relation*.

In this paper we will consider relations representing strict preference, indifference and possibly weak preference, respectively, denoted as P, I, Q. Relation Q is expected to represent a situation of hesitation between preference and indifference. The reason for which such a relation can be interesting will be discussed in Section 4. Such relations are expected to satisfy some "natural" properties: I is reflexive and symmetric; P and Q are asymmetric; $I \cup P \cup Q$ is complete; P, Q and I are mutually exclusive.

A useful tool to study the (possibly minimal) numerical representation of a preference structure is the potential function in a valued graph. Let $G = (A, U, v)$ be a valued graph on a finite set of nodes A; a real value $v(a, b)$ is attached to each arc (a, b) of U.

Definition 2.2. A potential function of the valued graph $G = (A, U, v)$ is a function $g : A \mapsto \mathbb{R}$ such that, $\forall (a, b) \in U,\ g(a) \geqslant g(b) + v(a, b)$.

It is easy to see that if g is a potential function whose minimal value is 0, then $g(a)$ cannot be smaller than the maximal value of the paths starting from a. A fundamental result is the following [9].

Theorem 2.3. *A valued graph admits potential functions iff there is no circuit of strictly positive value in the graph. The smallest non-negative potential function assigns to each node the maximal value of the paths starting from the node*.

3. Interval orders

Definition 3.1. A $\langle P, I\rangle$ preference structure on a finite set A is an interval order iff $\exists l, r : A \mapsto \mathbb{R}^+$ such that, $\forall x, y \in A$:

(i) $r(x) \geqslant l(x)$;
(ii) $P(x, y) \Leftrightarrow l(x) > r(y)$;
(iii) $I(x, y) \Leftrightarrow l(x) \leqslant r(y)$ and $l(y) \leqslant r(x)$.

Any couple (l, r) satisfying the above conditions is a general representation of the interval order. Since A is finite, given a general representation (l, r) of an interval order, there exists a positive constant $\varepsilon = \min_{(a,b)\in P}\{l(a) - r(b)\}$. The triple (l, r, ε) is called an ε-representation of the interval order. With an ε-representation, condition (ii) of Definition 3.1 can be rewritten as: $P(x, y) \Leftrightarrow l(x) \geqslant r(y) + \varepsilon$. Among all the possible ε-representations (with the same ε), the minimal one is of special interest. Naturally, it is defined as an ε-representation (l^*, r^*, ε) satisfying, for any other ε-representation (l, r, ε) $\quad \forall a \in A,\ l^*(a) \leqslant l(a) \quad$ and $\quad r^*(a) \leqslant r(a)$. The construction of the minimal representation is based on the following results.

Theorem 3.1. *Let $\langle P, I\rangle$ be an interval order on a finite set A, and let $T_l = P.I$, $T_r = I.P$. Then*

(i) *T_l, T_r are weak orders on A;*
(ii) *$T_l^{\sim}, T_r^{\sim}$ are equivalence relations and T_l, T_r are linear orders on $A/T_l^{\sim}, A/T_r^{\sim}$;*
(iii) *If $(a, b) \in T_l^{\sim} \cap T_r^{\sim}$ then there exists (l, r)* s.t. $l(a) = l(b) \wedge r(a) = r(b)$.

Proof. See Fishburn [2, Theorem 2, Chapter 2, p. 22]. □

Let us now define two copies of A, say A_l, and A_r. We define T_0 on $A_l \cup A_r$ as follows:

$T_0(a_l, b_l) \Leftrightarrow T_l(a, b)$;
$T_0(a_r, b_r) \Leftrightarrow T_r(a, b)$;
$T_0(a_l, b_r) \Leftrightarrow P(a, b)$;
$T_0(a_r, b_l) \Leftrightarrow I(a, b)$ or $P(a, b)$.

Theorem 3.2. *Let $\langle P, I\rangle$ be an interval order on a finite set A, and let T_l, T_r, T_0 defined as above. Then*

(i) *T_0 is a weak order on $(A_l \cup A_r)$;*
(ii) *$T_0^{\sim}$ is an equivalence relation and T_0 is a linear order on $(A_l \cup A_r)/T_0^{\sim}$;*
(iii) *$(A_l \cup A_r)/T_0^{\sim} = (A_l/T_l^{\sim}) \cup (A_r/T_r^{\sim})$;*
(iv) *$x \in A_l/T_l^{\sim} \Rightarrow T_0(y, x)$ for some $y \in A_r/T_r^{\sim}$; $y \in A_r/T_r^{\sim} \Rightarrow T_0(y, x)$ for some $x \in A_l/T_l^{\sim}$,*
$T_0(x_1, x_2),\ x_1, x_2 \in A_l/T_l^{\sim} \Rightarrow x_1 T_0 y T_0 x_2$ for some $y \in A_r/T_r^{\sim}$, $T_0(y_1, y_2),\ y_1, y_2 \in A_r/T_r^{\sim} \Rightarrow y_1 T_0 x T_0 y_2$ for some $x \in A_l/T_l^{\sim}$.

Proof. See Fishburn [2, Theorem 3, Chapter 2, p. 23]. □

T_l (T_r) represents the order of the left (right) endpoints of the intervals associated to elements of A. Each equivalence class in $A/T_l^{\sim}$, $(A/T_r^{\sim})$ represents a group of elements whose left (right) endpoints can be identical. T_0 represents the order of all such endpoints. Theorem 3.2 shows that after a class of left endpoints there is a class of right endpoints followed by a class of left endpoints and so on.

Theorem 3.3. *Let $\langle P, I\rangle$ be an interval order on a finite set A, and T_l, T_r, T_0 defined as above, then*

(i) *$A/T_l^{\sim}$ and $A/T_r^{\sim}$ have the same cardinality, say m;*

(ii) *If* $A/T_l^{\sim} = \{A_m T_0 A_{m-1} \dots T_0 A_1\}$ *and* $A/T_r^{\sim} = \{B_m T_0 B_{m-1} \dots T_0 B_1\}$ *then* $(A_l \cup A_r/T_0^{\sim}) = \{B_m, A_m, \dots, B_1, A_1\}$, *and* $B_m T_0 A_m T_0 B_{m-1} T_0 A_{m-1} \dots T_0 B_1 T_0 A_1$.

Proof. See Fishburn [2, Theorem 5, Chapter 2, p. 26]. □

The construction of the minimal ε-representation of an interval order is straightforward from Theorems 2.3 and 3.3. The number *m* is called magnitude of the interval order. With $\varepsilon = 1$, the minimal 1-representation is a representation on the smallest possible interval of the set of integer numbers.

4. PQI interval orders (PQI–IO)

As already discussed in Fishburn [3], interval orders, such as presented in the above section, are not the only way to consider the comparison of objects represented by intervals. However, the alternatives considered in the literature (see [3]) are all based on the hypothesis that only strict preference and indifference can be considered. The different preference structures just consider different ways to separate the two relations.

The comparison of intervals, however, allows to consider a third relation, namely a relation representing hesitation between strict preference and indifference. Vincke [14] discussed and characterised a preference structure with such a hypothesis. In that case the hesitation was due to the presence of two thresholds (intervals with an intermediate point). Another way to let appear such an hesitation is to consider that when two intervals have a non-empty intersection, but one is "more to the right" (in the sense of the reals) there exist reasons for which a preference can be established (for a discussion on this point see also Tsoukiàs et al. [10]. Such a preference structure, called *PQI*–IO has been characterised by Tsoukiàs and Vincke [12,13]. Further on, Ngo The et al. [5] showed that the satisfaction of the characteristic conditions of a *PQI*–IO is polynomial.

The open problem is that such results do not tell us how to obtain a numerical representation (possibly a minimal one), under form of intervals, for the elements of a set *A* as soon as the theorem of existence of a *PQI*–IO is demonstrated. Thus, we do not know if this is an "easy" problem or not. In this section we extend Fishburn [2] results in the case of *PQI*–IO. Practically, we show that it is possible to organise the intervals (which have to exist) in such a way that classes of left endpoints are followed by classes of right endpoints and so on. With such a result it is possible to establish "easy" algorithms enabling to define the numerical representation (possibly minimal) for a given *PQI*–IO. First, we recall some definitions and fundamental results concerning *PQI*–IO.

Definition 4.1. A $\langle P, Q, I\rangle$ preference structure on a finite set *A* is a *PQI*–IO iff $\exists : l, r : A \mapsto \mathbb{R}^+$, such that $\forall x, y \in A$:

(i) $r(x) \geqslant l(x)$;
(ii) $P(x, y) \Leftrightarrow l(x) > r(y)$;
(iii) $Q(x, y) \Leftrightarrow r(x) > r(y) \geqslant l(x) > l(y)$;
(iv) $I(x, y) \Leftrightarrow r(x) \geqslant r(y) \geqslant l(y) \geqslant l(x)$ or $r(y) \geqslant r(x) \geqslant l(x) \geqslant l(y)$.

A couple (l, r) satisfying these conditions is a general representation of the *PQI*–IO.

Theorem 4.1. *A* $\langle P, Q, I\rangle$ *preference structure on a finite set A is a PQI*–IO *iff there exists a partial order L such that*:

(i) $I = L \cup R \cup I_d$ *where* $I_d = \{(x, x), x \in A\}$ *and* $R = L^{-1}$;
(ii) $(P \cup Q \cup L).P \subset P$;
(iii) $P.(P \cup Q \cup R) \subset P$;
(iv) $(P \cup Q \cup L).Q \subset P \cup Q \cup L$;
(v) $Q.(P \cup Q \cup R) \subset P \cup Q \cup R$.

Proof. See Tsoukiàs and Vincke [13]. □

An algorithm to detect a *PQI*–IO, i.e. to construct *L*, was presented in Ngo The et al. [5]. Since *A* is finite, there exists

$$\varepsilon = \min\{|x - y|, x, y \in \{l(a), a \in A\} \cup \{r(a), a \in A\}\}.$$

The triple (l, r, ε) is called an ε-representation of the *PQI*–IO. With an ε-representation, conditions (ii) and (iii) of Definition 4.1 become: $P(x, y) \Leftrightarrow l(x) \geqslant l(y) + \varepsilon$ and $Q(x, y) \Leftrightarrow r(x) \geqslant r(y) + \varepsilon$ and $r(y) \geqslant l(x) \geqslant l(y) + \varepsilon$.

The problem to face now is the construction of a (possibly minimal) numerical representation of a *PQI*–IO. Imagine the following situation: a decision maker comes up with some preference statements expressed on a set of alternatives. Such preferences include situations of hesitation for some pairs of alternatives. A first task for the analyst could be to check whether the hesitation of the decision maker could be modelled associating intervals to the alternatives. For this purpose (s)he might use the results in Tsoukiàs and Vincke [12,13] and in Ngo The et al. [5] and check if the conditions of existence of a *PQI*–IO are satisfied. Suppose it is the case. The problem now is to suggest to the decision maker the numerical representation of such intervals. Such a task does not has an intuitive answer and can represent several difficulties as can be seen from the following example.

Example 4.1. Consider the case of three alternatives and the following preferences expressed on them: *aQb*, *aIc*, *bIc*;

	a	b	c
a		Q	I
b			I
c			

It is easy to check that such preferences can be represented as a *PQI*–IO. However, it is also easy to verify that there exist two completely different relations *L* satisfying the Theorem 4.1 each one admitting a 1-representation as can be seen in the tables besides:

	a	b	c
a		Q	L
b			L
c			

	a	b	c
a		Q	R
b			R
c			

	a	b	c
l_1	1	0	1
r_1	2	1	1

	a	b	c
l_2	1	0	0
r_2	2	1	2

If there was a minimal 1-representation l^*, r^* then $l^*(a) \leqslant \min\{l_1(a), l_2(a)\} = 1$. Similarly, $l^*(b) \leqslant 0$, $l^*(c) \leqslant 0$, $r^*(a) \leqslant 2$, $r^*(b) \leqslant 1$, $r^*(c) \leqslant 1$. Furthermore, $aQb \Rightarrow [(r^*(a) \geqslant r^*(b) + 1) \wedge (r^*(b) \geqslant l^*(a) \geqslant l^*(b) + 1)] \Rightarrow (r^*(a) \geqslant 2) \wedge (l^*(a) \geqslant 1) \wedge (r^*(b) \geqslant 1)$. We have then $l^*(a) = 1, r^*(a) = 2, l^*(b) = 0, r^*(b) = 1, l^*(c) = 0, r^*(c) \leqslant 1$ and $r^*(c)$ must be either 0 or 1; neither of these values is acceptable.

This example shows that the notion of minimal representation does not make sense for a *PQI*–IO. Therefore, it is necessary to limit the question concerning the (possibly minimal) numerical representation to an instance of a *PQI*–IO corresponding to a specific relation L. We call such an instance a "separated *PQI*–IO". The relations to consider in a separated *PQI*–IO are P, Q, L, I_d. For the rest of the paper we are going to consider only such "separated *PQI*–IO". The ε-representation (l, r, ε) of a separated *PQI*–IO is defined in the same way as the one of a *PQI*–IO. [1]

Let us now begin with the following result presenting the IO associated to a separated *PQI*–IO through the reduction of the relations I_d, L, Q into $\hat{I}$.

Theorem 4.2. *If $\langle P, Q, L, I_d\rangle$ is a separated PQI–*IO *and $\hat{I} = I_d \cup L \cup L^{-1} \cup Q \cup Q^{-1}$ then $\langle P, \hat{I}\rangle$ is an* IO.

Proof. See Tsoukiàs and Vincke [13]. □

Let us define the following relations: $\hat{T}_l = P.\hat{I}$; $\hat{T}_r = \hat{I}.P$;

We introduce two copies of A, say A_l and A_r and we construct the relation $\hat{T}_0$ on $A_l \cup A_r$ as follows:

$\hat{T}_0(a_l, b_l) \Leftrightarrow \hat{T}_l(a, b)$,
$\hat{T}_0(a_r, b_r) \Leftrightarrow \hat{T}_r(a, b)$,

[1] There is one point we would like to make clear about our choice of "separated *PQI*–IO" to deal with. The non-existence of the minimal representation is not the only reason. Suppose that the decision maker has a *PQI*–IO and wants just one numerical representation, not necessarily the minimal one. Can we provide an algorithm to produce such a representation directly from the three relations P, Q, I? As far as we know, the answer is no. We cannot determine a representation of a *PQI*–IO without knowing a priori that this structure is a *PQI*–IO. Therefore, the question makes sense only if there is an algorithm that can prove the existence of the relation L without explicitly constructing one, but the only way (we know) is to explicitly construct the relation L. With our current knowledge, we have to use the algorithm in Ngo The et al. [5] to verify if the structure is an *PQI*–IO. If the answer is yes, the algorithm provides L. With this relation L, we can determine a numerical representation of the "separated *PQI*–IO'. This is also a representation of the original structure *PQI*–IO.

Table 1
Example of a separated *PQI*–IO

	a	b	c	d	e	f	g	h	a	b	c	d	e	f	g	h
a		P	P	P	P	P	P	P		P	P	P	P	P	P	P
b			Q	P	P	P	P	P			$\hat{T}_l$	P	P	P	P	P
c				Q	P	P	P	Q				$\hat{T}_r$	P	P	P	$\hat{T}_l, \hat{T}_r$
d					P	P	P	L					P	P	P	$\hat{T}_l$
e						Q	P	L						$\hat{T}_l$	P	$\hat{T}_l, \hat{T}_r^{-1}$
f							Q	L							$\hat{T}_r$	$\hat{T}_r^{-1}$
g								L								$\hat{T}_r^{-1}$
h																

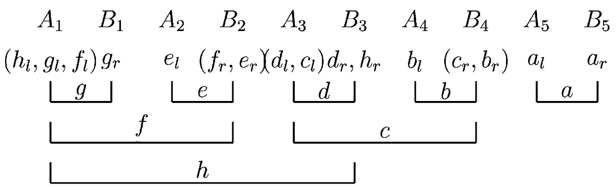

Fig. 2. The intervals associated to the interval order of Example 4.2.

$\hat{T}_0(a_l, b_r) \Leftrightarrow P(a, b),$
$\hat{T}_0(a_r, b_l) \Leftrightarrow \neg P(b, a).$

Since $\langle P, \hat{I}\rangle$ is an interval order, we can apply Theorems 3.1, 3.2, and 3.3 for the relations $\hat{T}_l, \hat{T}_r, \hat{T}_0$. We obtain: $m = |A_l/\hat{T}_l^{\sim}| = |A_r/\hat{T}_r^{\sim}|$ the magnitude of the interval order $\langle P, \hat{I}\rangle$;

$(A_l \cup A_r)/\hat{T}_0^{\sim} = (A_l/\hat{T}_l^{\sim}) \cup (A_r/\hat{T}_r^{\sim});$
$A_l/\hat{T}_l^{\sim} = \{A_m \hat{T}_0 A_{m-1} \hat{T}_0 \dots A_1\};$
$A_r/\hat{T}_r^{\sim} = \{B_m \hat{T}_0 B_{m-1} \hat{T}_0 \dots B_1\};$
$B_m \hat{T}_0 A_m \hat{T}_0 B_{m-1} \dots \hat{T}_0 B_1 \hat{T}_0 A_1.$

Example 4.2. Consider the information in Table 1 where the left part presents a separated *PQI*–IO. The right part resumes the relations $\hat{T}_l, \hat{T}_r$. Since $aPb \Rightarrow a\hat{T}_l b \wedge a\hat{T}_r b$, there is no need to write $\hat{T}_l, \hat{T}_r$ when P is the case.

The reader can check easily that $b\hat{T}_l c$ holds since $bPh\hat{I}c$ holds and so on. Considering the interval order $\langle P, \hat{I}\rangle$ and applying the theorems of Section 3 we have $m=5$, $A/\hat{T}_l^{\sim} = \{A_1 = \{h_l, g_l, f_l\}, A_2 = \{e_l\}, A_3 = \{d_l, c_l\}, A_4 = \{b_l\}, A_5 = \{a_l\}\}$ and $A/\hat{T}_r^{\sim} = \{B_1 = \{g_r\}, B_2 = \{f_r, e_r\}, B_3 = \{d_r, h_r\}, B_4 = \{c_r, b_r\}, B_5 = \{a_r\}\}$. Such a numerical representation is shown in Fig. 2.

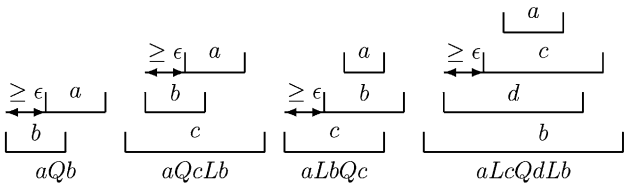

Fig. 3. Cases of $l(a) \neq l(b)$ due to Q in a separated *PQI*–IO.

We extend now the relations $\hat{T}_l, \hat{T}_r, \hat{T}_0$ into T_l, T_r, T_0 as follows:

$Q_l = Q \cup L.Q \cup Q.L \cup L.Q.L$; $Q_r = Q \cup R.Q \cup Q.R \cup R.Q.R$;
$T_l = \hat{T}_l \cup Q_l$; $T_r = \hat{T}_r \cup Q_r$;
$T_0(a_l, b_l) \Leftrightarrow T_l(a, b)$, $T_0(a_r, b_r) \Leftrightarrow T_r(a, b)$, $T_0(a_l, b_r) \Leftrightarrow P(a, b)$,
$T_0(a_r, b_l) \Leftrightarrow \neg P(b, a)$. It is obvious that $\hat{T}_0 \subset T_0$, as $\hat{T}_l \subset T_l$ and $\hat{T}_r \subset T_r$.

The idea behind the construction of Q_l, Q_r, T_l, T_r and T_0 is the following. The relations T_l, T_r, T_0 play the same role as that of their counterparts in an IO. In fact, when $(a, b) \in I$ in an IO we cannot say whether $l(a), l(b)$ (the left endpoints of the intervals representing a, b) can be unified ($l(a) = l(b)$). The role of the relation $T_l = P.I$ is to identify all the cases where $l(a) \neq l(b)$. The same approach is used in the case of a separated *PQI*–IO. We use $\hat{T}_l$ to identify cases where $l(a) \neq l(b)$ due to P (through the use of the associated IO). However, in the case of a *PQI*–IO this is not sufficient. There might be cases where $l(a) \neq l(b)$ because of Q. For this purpose we use Q_l. The four components of Q_l are illustrated in Fig. 3. The relation T_0 reflects the order of all the endpoints and its construction from T_l, T_r is the same in the two structures.

After having constructed the relations helping us to determine the arrangement of the endpoints, we try now to extend Theorems 3.1, 3.2, and 3.3 using T_l, T_r, T_0.

Proposition 1. *Let* $\langle P, Q, L, I_d \rangle$ *be a separated PQI*–IO, *then*

(i) $Q.L \subset Q \cup L$ *and* $R.Q \subset R \cup Q$;
(ii) $P.L \subset P \cup Q \cup L$ *and* $R.P \subset P \cup Q \cup R$;
(iii) $P.Q^{-1} \subset (P \cup Q \cup L)$ *and* $Q^{-1}.P \subset (P \cup Q \cup R)$;
(iv) $Q_l \cap \hat{T}_l^{-1} = Q_r \cap \hat{T}_r^{-1} = \emptyset$;
(v) $P \cup Q \subset T_l \subset L \cup P \cup Q$ *and* $P \cup Q \subset T_r \subset R \cup P \cup Q$;
(vi) $(P^{-1} \cup Q^{-1} \cup R) \subset \neg T_l \subset (P^{-1} \cup Q^{-1} \cup L \cup R)$, *and* $(P^{-1} \cup Q^{-1} \cup L) \subset \neg T_r \subset (P^{-1} \cup Q^{-1} \cup L \cup R)$.
(vii) $T_l.P \subset P$ *and* $P.T_r \subset P$
(viii) $P.T_l \subset T_l$ *and* $T_r.P \subset T_r$

Proof. See Appendix A. □

Theorem 4.3. *Let* $\langle P, Q, L, Id \rangle$ *be a separated PQI*–IO, *then*

(i) T_l, T_r *are weak orders on A*;

(ii) $T_l^{\sim}, T_r^{\sim}$ *are equivalence relations*; T_l, T_r *are linear orders on* $A/T_l^{\sim}, A/T_r^{\sim}$;
(iii) *If* $(a, b) \in T_l^{\sim} \cap T_r^{\sim}$ *then there exists* (l, r) *s.t.* $l(a) = l(b) \ \wedge \ r(a) = r(b)$.
(iv) $\forall a \in A : [a]_{T_l^{\sim}} \subset [a]_{\hat{T}_l^{\sim}}$ *and* $[a]_{T_r^{\sim}} \subset [a]_{\hat{T}_r^{\sim}}$.

Proof. See Appendix A. □

This result is the generalisation of Theorem 3.1 showing the grouping of all left (right) endpoints by T_l (T_r). Condition (iv) shows that T_l (T_r) is an extension of $\hat{T}_l$ ($\hat{T}_r$) and, consequently, $T_l^{\sim}$ ($T_r^{\sim}$) is a refinement of $\hat{T}_l^{\sim}$ ($\hat{T}_r^{\sim}$).

Theorem 4.4. *Let* $\langle P, Q, L, I_d \rangle$ *be a separated PQI–*IO, *then*

(i) T_0 *is a weak order on* $(A_l \cup A_r)$;
(ii) $T_0^{\sim}$ *is an equivalence relation and* T_0 *is a linear order on* $(A_l \cup A_r)/T_0^{\sim}$;
(iii) $(A_l \cup A_r)/T_0^{\sim} = (A_l/T_l^{\sim}) \cup (A_r/T_r^{\sim})$;

Proof. See Appendix A. □

This result extends Theorem 3.2. The only difference concerns property (iv) of Theorem 3.2. In an IO, two consecutive left (right) endpoints can always be unified (we can give them the same value). Therefore, all consecutive left (right) endpoints form a left (right) group. Thus, we obtain an alternation of left and right groups. This is not any more true if $l(a), l(b)$ are two consecutive left endpoints in a separated *PQI*–IO. There might be several possible inequalities between *a* and *b*. For example, if $(a, b) \in Q_l = Q \cup L.Q \cup Q.L \cup L.Q.L$, then $l(a)$ must be $\geqslant l(b) + \varepsilon$ and they cannot be unified. They belong to different groups (classes of equivalence of $T_l^{\sim}$). The following theorem shows how groups of left (right) endpoints can be defined.

Theorem 4.5. *Let* $\langle P, Q, I \rangle$ *be a separated PQI–*IO, *and* $m = |A/\hat{T}_l^{\sim}|$, $l = |A/T_l^{\sim}|$, $r = |A/T_r^{\sim}|$, $A/\hat{T}_l^{\sim} = \{A_i, i = 1..m\}$, $A/\hat{T}_r^{\sim} = \{B_i, i = 1, \ldots, m\}$, *then*

(i) *classes of* $A_l/T_l^{\sim}, A_r/T_r^{\sim}$ *can be arranged in such a way that*

$$A_l/T_l^{\sim} = \{\underbrace{X_l T_0 X_{l-1} T_0 \ldots X_{l_1}}_{A_m} T_0 \underbrace{X_{l_1-1} T_0 X_{l_1-2} T_0 \ldots X_{l_2}}_{A_m-1} \ldots$$

$$\underbrace{X_{l_{m-1}-1} T_0 X_{l_{m-1}-2} T_0 \ldots X_1}_{A_1}\},$$

$$A_r/T_r^{\sim} = \{\underbrace{Y_r T_0 Y_{r-1} T_0 \ldots Y_{r_1}}_{B_m} T_0 \underbrace{Y_{r_1-1} T_0 Y_{r_1-2} T_0 \ldots Y_{r_2}}_{B_m-1} \ldots$$

$$\underbrace{Y_{r_{m-1}-1} T_0 Y_{r_{m-1}-2} T_0 \ldots Y_1}_{B_1}\};$$

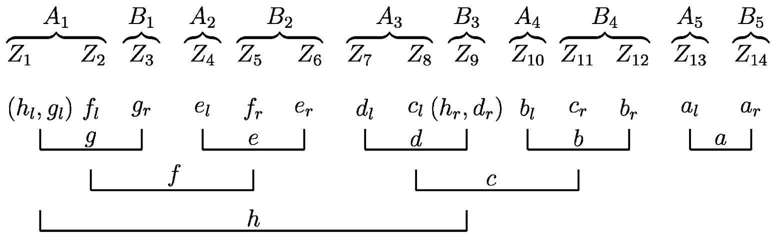

Fig. 4. Separation of the intervals of Example 4.2.

(ii) *with this arrangement, the linear order* T_0 *on* $(A_l \cup A_r)/T_0^{\sim}$ *becomes*:

$$\underbrace{Y_r T_0 Y_{r-1} \ldots Y_{r_1}}_{B_m} T_0 \underbrace{X_l T_0 X_{l-1} T_0 \ldots X_{l_1}}_{A_m} T_0 \ldots$$
$$\underbrace{Y_{r_{m-1}-1} T_0 Y_{r_{m-1}-2} T_0 \ldots Y_1}_{B_1} T_0 \underbrace{X_{l_{m-1}-1} T_0 X_{l_{m-1}-2} T_0 \ldots X_1}_{A_1}.$$

Proof. (i) Immediate from $\forall a \in A$, $[a]_{T_l^{\sim}} \subset [a]_{\hat{T}_l^{\sim}}$, $T_l \cup T_r \subset T_0$.

(ii) Immediate from (i) and Theorem 3.3. $\square$

Like its counterpart (Theorem 3.3), this result represents the grouping of the endpoints in classes of equivalence (of $T_0^{\sim}$) and their arrangement. There are two grouping levels. The first one is due to the IO $\{P, \hat{I}\}$ with m left groups $(A_1, A_2, \ldots, A_m)$ and m right groups $(B_1, B_2, \ldots, B_m)$. The second one, finer, is due to the extended relation T_l, T_r. Each group of endpoints at this level (X_k or Y_l) is a subset of a group A_i (B_j). We can now arrange all the elements of $(A_l \cup A_r)/T_0^{\sim}$ according to the linear order T_0 and then label them by Z_i where the index i is the rank of the group in T_0. We have then $Z_{l+r} T_0 Z_{l+r-1} T_0, \ldots, Z_1$. In other terms in order to fix the intervals of a separated *PQI*–IO we first separate relation P, thus obtaining a first group of endpoints and then we refine each of such groups using relation Q and L. $M = l + r - m$ is called the magnitude of the separated *PQI*–IO. It is easy to verify that when $l = r = m$ then $Q = \emptyset$, the preference structure in question is an IO with magnitude m.

4.1. Continuation of Example 4.2

We have $l = 7$, $r = 7$, $M = l + r - m = 9$. *After the re-arrangement, we obtain the following groups* (*see the* Fig. 4).

$Z_1 = X_1 = \{h_l, g_l\}$, $Z_2 = X_2 = \{f_l\}$, $Z_3 = Y_1 = \{g_r\}$, $Z_4 = X_3 = \{e_l\}$, $Z_5 = Y_2 = \{f_r\}$, $Z_6 = Y_3 = \{e_r\}$, $Z_7 = X_4 = \{d_l\}$, $Z_8 = X_5 = \{c_l\}$, $Z_9 = Y_4 = \{d_r, h_r\}$, $Z_{10} = X_6 = \{b_l\}$, $Z_{11} = Y_5 = \{c_r\}$, $Z_{12} = Y_6 = \{b_r\}$, $Z_{13} = X_7 = \{a_l\}$, $Z_{14} = Y_7 = \{a_r\}$.

The two grouping levels are:

$$\underbrace{Z_{14}}_{B_5}\,T_0\,\underbrace{Z_{13}}_{A_5}\,T_0\,\underbrace{Z_{12}T_0Z_{11}}_{B_4}\,T_0\,\underbrace{Z_{10}}_{A_4}\,T_0\,\underbrace{Z_9}_{B_3}\,T_0\,\underbrace{Z_8T_0Z_7}_{A_3},$$
$$T_0\,\underbrace{Z_6T_0Z_5}_{B_2}\,T_0\,\underbrace{Z_4}_{A_2}\,T_0\,\underbrace{Z_3}_{B_1}\,T_0\,\underbrace{Z_2T_0Z_1}_{A_1}.$$

The relation between T_0 and any ε-representation is shown in the following proposition. This result is used for the construction of a minimal representation as can be seen from the following two results.

Proposition 2. *Let* (l, r, ε) *be an* ε*-representation of a separated PQI*–IO, *then*:

(i) $T_0(a_l, b_l) \Rightarrow l(a) \geqslant l(b) + \varepsilon$;
(ii) $T_0(a_r, b_r) \Rightarrow r(a) \geqslant r(b) + \varepsilon$;
(iii) $T_0(a_l, b_r) \Rightarrow l(a) \geqslant r(b) + \varepsilon$;
(iv) $T_0(a_r, b_l) \Rightarrow r(a) \geqslant l(b)$;

Proof. See Appendix A.

Theorem 4.6. *Given a separated PQI*–IO *and a positive constant* ε, *let define* $l^*(a) = (i - j + 1)\varepsilon$ *where* $a_l \in Z_i \subset A_j$; $r^*(a) = (i - j)\varepsilon$ *where* $a_r \in Z_i \subset B_j$; *where* A_j, B_j, Z_i *defined in Theorem* 4.5. *Then* (l^*, r^*, ε) *is its minimal* ε*-representation* (l^* *and* r^* *are integral multiples of* ε).

Proof. See Appendix A.

4.2. Continuation of Example 4.2

Applying Theorem 4.6 *we obtain the minimal* 1*-representation as following*:

	l	Explication	r	Explication
a	8	$a_l \in Z_{13} \subset A_5(8 = 13 - 5)$	8	$a_r \in Z_{14} \subset B_5(8 = 14 - 5 - 1)$
b	6	$b_l \in Z_{10} \subset A_4(6 = 10 - 4)$	7	$b_r \in Z_{12} \subset B_4(7 = 12 - 4 - 1)$
c	5	$c_l \in Z_8 \subset A_3(5 = 8 - 3)$	6	$c_r \in Z_{11} \subset B_4(6 = 11 - 4 - 1)$
d	4	$d_l \in Z_7 \subset A_3(4 = 7 - 3)$	5	$d_r \in Z_9 \subset B_3(5 = 9 - 3 - 1)$
e	2	$e_l \in Z_4 \subset A_2(2 = 4 - 2)$	3	$e_r \in Z_6 \subset B_2(3 = 6 - 2 - 1)$
f	1	$f_l \in Z_2 \subset A_1(1 = 2 - 1)$	2	$f_r \in Z_5 \subset B_2(2 = 5 - 2 - 1)$
g	0	$g_l \in Z_1 \subset A_1(0 = 1 - 1)$	1	$g_r \in Z_3 \subset B_1(1 = 3 - 1 - 1)$
h	0	$h_l \in Z_1 \subset A_1(0 = 1 - 1)$	5	$h_r \in Z_9 \subset B_3(5 = 9 - 3 - 1)$

Let us resume our findings. Proposition 1 and Theorems 4.3 and 4.4 show that it is possible, given a *PQI* interval order on a set *A*, to obtain two weak orders on *A*, named T_l and T_r, which represent the ordering of the left and right endpoints, respectively, of the intervals associated to each element of A. Moreover, using Theorem 4.5, we show that it is possible to define a linear order T_0 by which left and right endpoints are grouped into classes which are ordered alternatively by T_0. Proposition 2 and Theorem 4.6 show that, given a separated *PQI*–IO, there always exists an ε-minimal representation, ε being a positive constant. Such results show that the intervals that can be associated to a *PQI*–IO "behave" as the ones that can be associated to an IO. Thus, in order to obtain a numerical representation of a *PQI*–IO we need to arrange elements in A in such a way to define a sequence of left–right endpoints each separated by at least an ε.

5. Algorithms

A straightforward application of the above results in order to determine a minimal ε-representation of a *PQI*–IO is rather complicated as it requires the explicit determination of $\hat{T}_l, \hat{T}_r, T_l, T_r, T_0, (A_l \cup A_r)/T_0^{\sim} \ldots$. In this section, we present more results allowing to determine first a numerical representation and second a minimal ε-representation using two algorithms. The first algorithm (in $\mathrm{O}(n^2)$) determines a representation where all endpoints are distinct. The endpoints which could be identical will be unified in the second algorithm (in $\mathrm{O}(n)$) to obtain a minimal ε-representation.

Proposition 3. *Let $\langle P, Q, L, I_d\rangle$ be a separated PQI*–IO, (l, r, ε) *be a representation in which all endpoints are distinct, $B = \{l(x), r(x), x \in A\}$ be the set of all values of the representation. Let us define the relation T on $(A_l \cup A_r)$ as*:

$T(a_r, a_l)$;
$T(a_l, b_l) \Leftrightarrow P(a, b)\ or\ Q(a, b)\ or\ L(a, b)$;
$T(a_r, b_r) \Leftrightarrow P(a, b)\ or\ Q(a, b)\ or\ R(a, b)$;
$T(a_l, b_r) \Leftrightarrow P(a, b)$; $T(a_r, b_l) \Leftrightarrow \neg P(b, a)$. *Then*:
(i) $T_0 \subset T$, *i.e. T is an extension of T_0.*
(ii) $(A_l \cup A_r, T)$ *is a linear order and an isomorphism of the order* $(B, >)$.

Proof. (i) $(x, y) \in T_0$. If $x = a_l, y = b_l$ then $(a, b) \in T_l \subset P \cup Q \cup R$ then $T(x, y)$. The same argument for $x = a_r, y = b_r$. By construction of T and T_0, if $x = a_l, y = b_r$ or $x = a_r, y = b_l$ then $T(x, y)$.

(ii) Obviously $(B, >)$ is a linear order as $l(x), r(x)$ have all distinct values. With the mapping $\phi : A_l \cup A_r \mapsto B$ defined as: $\phi(a_l) = l(a)$, $\phi(a_r) = r(a)$, it is easy to check that ϕ is a bijection and $T(x, y) \Leftrightarrow \phi(x) > \phi(y)$. $\square$

We can consider now the valued graph $(A_l \cup A_r, T, v)$ where $v(x, y) = \varepsilon, \forall x, y \in A$. It is obvious that $(l(a) = \varepsilon g(a_l), r(a) = \varepsilon g(a_r), \varepsilon)$, where $g(x)$ is the rank of x in the linear order T (starting with 0), is a minimal ε-representation with distinct endpoints. From Proposition 3, we have:

$\forall a_l \in A_l : T^+(a_l) = \{x_l, x_r : P(a, x), x \in A\} \cup \{x_l : Q(a, x), x \in A\} \cup \{x_l : L(a, x), x \in A\}$;

$\forall a_r \in A_r : T^+(a_r) = \{a_l, x \in A\} \cup \{x_l, x_r : P(a,x), x \in A\} \cup \{x_l, x_r : Q(a,x), x \in A\} \cup \{x_l : Q^{-1}(a,x), x \in A\} \cup \{x_l : L(a,x), x \in A\} \cup \{x_l, x_r : R(a,x), x \in A\}$.

This result leads us to the following formula:

$\forall a \in A,\ g(a_l) = |T^+(a_l)| = 2|P^+(a)| + |Q^+(a)| + |L^+(a)| + 1;$
$g(a_r) = |T^+(a_r)| + 1 = 1 + 2|P^+(a)| + 2|Q^+(a)| + |Q^{-1+}| + |L^+(a)| + 2|R^+(a)|.$

The function g can be implemented using the following algorithm ($O(n^2)$):

```
n = |A|, fl[1..n], fr[1..n]/ * g(a_l), g(a_r) * /
M[1..n,1..n]; /* matrix representing P,Q,L*/
procedure numerical_representation
     forall i fl[i] = 0, fr[i] = 1
     endfor
     forall i, j, j > i, switch (M[i,j])
          case P:
               fl[i] = fl[i]+2
               fr[i] = fr[i]+2
          case P^-1:
               fl[j] = fl[j]+2
               fr[j] = fr[j]+2
          case Q:
               fl[i] = fl[i]+1
               fr[i] = fr[i]+2
               fr[j] = fr[j]+1
          case Q^-1:
               fl[j] = fl[j]+1
               fr[j] = fr[j]+2
               fr[i] = fr[i]+1
          case L:
               fl[i] = fl[i]+1
               fr[i] = fr[i]+1
               fr[j] = fr[j]+2
          case R:
               fl[j] = fl[j]+1
               fr[j] = fr[j]+1
               fr[i] = fr[i]+2
     endswitch
endfor
```

5.1. Continuation of Example 4.2

We apply the algorithm to the data of our example and we verify that the result is compatible with Fig. 4.

$g(x_l) = 2|P^+| + |Q^+| + |L^+|,\ g(x_r) = 1 + 2|P^+| + 2|Q^+| + |Q^{-1+}| + |L^+| + 2|R^+|$

x	$g(x_l)$	$g(x_r)$
a	$14 = 2 \times 7 + 0 + 0$	$15 = 1 + 2 \times 7 + 2 \times 0 + 0 + 0 + 2 \times 0$
b	$11 = 2 \times 5 + 1 + 0$	$13 = 1 + 2 \times 5 + 2 \times 1 + 0 + 0 + 2 \times 0$
c	$8 = 2 \times 3 + 2 + 0$	$12 = 1 + 2 \times 3 + 2 \times 2 + 1 + 0 + 2 \times 0$
d	$7 = 2 \times 3 + 0 + 1$	$9 = 1 + 2 \times 3 + 2 \times 0 + 1 + 1 + 2 \times 0$
e	$4 = 2 \times 1 + 1 + 1$	$6 = 1 + 2 \times 1 + 2 \times 1 + 0 + 1 + 2 \times 0$
f	$2 = 2 \times 0 + 1 + 1$	$5 = 1 + 2 \times 0 + 2 \times 1 + 1 + 1 + 2 \times 0$
g	$1 = 2 \times 0 + 0 + 1$	$3 = 1 + 2 \times 0 + 2 \times 0 + 1 + 1 + 2 \times 0$
h	$0 = 2 \times 0 + 0 + 0$	$10 = 1 + 2 \times 0 + 2 \times 0 + 1 + 0 + 2 \times 4$

Let us work on the minimal representation. By definition, $T_0 \subset T$, i.e., T is an extension of T_0, furthermore, this extension adds only pairs of either type $T(a_l, b_l)$ or $T(a_r, b_r)$ to T_0. We have seen in the previous section that the minimal ε-representation is based on T_0. The unification of endpoints is indeed a reduction from T to T_0: two consecutive left (right) end points (in T) which are not related by T_0 can be unified. Two consecutive endpoints $a_r T b_l$ can always be unified because $T_0(a_r, b_l)$ requires only $r(a) \geqslant l(b)$.

Proposition 4. *Let $\langle P, Q, L, I_d \rangle$ be a separated PQI–*IO*, then*:

(i) *if $a_l T b_l$ are two consecutive endpoints and $T_0(a_l, b_l)$ then $Q(a, b)$;*
(ii) *if $a_r T b_r$ are two consecutive endpoints and $T_0(a_r, b_r)$ then $Q(a, b)$.*

Proof. (i) If $(a_l, b_l) \in T_0$ then $(a, b) \in T_l = P.\hat{I} \cup Q \cup L.Q \cup Q.L \cup L.Q.L$. With the exception of Q, there is always at least an endpoint x such that $a_l T x T b_l$, i.e., a_l, b_l are not consecutive. For example, $(a, b) \in L.Q$ then $\exists c \in A$, $aLcQb$, and we have $a_l T c_l T b_l$. The other cases are similar. (ii) Similar to (i). □

As a consequence, two consecutive endpoints xTy can be unified if, $\exists a, b \in A$ such that one of the following conditions is satisfied:

(1) $x = a_r$, $y = b_l$; (2) $x = a_l$, $y = b_l$ and $L(a, b)$; (3) $x = a_r$, $y = b_r$ and $R(a, b)$.

We obtain the following algorithm in O(n) to unify endpoints:

```
Rank[1..2n]; /* 1..2n rank of element x ∈ A_l ∪ A_r */
Id[1..2n]; /* identification of element x ∈ A */
LR[1..2n]; /* left endpoint, right endpoint*/
M[1..n,1..n]; /* matrix representing P, Q, L */
X = 0; /* number of unifications realised, to be subtracted
from the rank to obtain the minimal representation */
procedure minimal_numerical_representation
 for i = 1..2n do
  Rank[i] = Rank[i]-X;
```

```
if i = 2n then stop endif;
Rank[i] = Rank[i]-X;
if [LR[i] = left and LR[i+1] = left and M[Id[i+1],Id[i]] = L]
or [LR[i] = right and LR[i+1] = right and M[Id[i+1],Id[i]] = R]
or [LR[i] = left and LR[i+1] = right] then
 X = X+1;
 endif;
endfor;
```

5.2. Continuation of Example 4.2

Applying the above algorithm to our example we obtain the following table. The reader may note that the algorithm treats the endpoints in the ascending order of their ranks, i.e. *aTb* means that the rank of *a* is superior to that of *b*, therefore *b* will appear before *a*.

Id	$Rank$	X	$Rank - X$	$observation$
h_l	0	0	0	$l, l, L(g, h)$
g_l	1	1	0	
f_l	2	–	1	l, r
g_r	3	2	1	
e_l	4	–	2	l, r
f_r	5	3	2	
e_r	6	–	3	
d_l	7	–	4	
c_l	8	–	5	l, r
d_r	9	4	5	$r, r, R(a, d)$
h_r	10	5	5	
b_l	11	–	6	l, r
c_r	12	6	6	
b_r	13	–	7	
a_l	14	–	8	l, r
a_r	15	7	8	

6. Conclusion

In this paper, we try to extend some well-known results concerning the numerical representation of interval orders in the case of *PQI*–IO. Such preference structures appear when, while comparing intervals, it might be interesting to distinguish a situation of hesitation between "sure" preference (empty intersection of the two intervals) and "sure" indifference (one interval included in the other).

As we have shown that the problem of numerical representations of a *PQI*–IO does not make sense, we have to study the problem through an instance of a *PQI*–IO, i.e. a separated *PQI*–IO. The aim of this effort is to study the foundations under which is possible to construct a numerical representation of a separated *PQI*–IO as soon as it has been demonstrated that such a representation exists. Not surprisingly we are able to demonstrate that there exist two weak orders, one representing the order of the left endpoints and one representing the order of the right endpoints. On that basis is possible to construct a numerical representation.

In the paper we demonstrate the theorems which enable to show what the numerical representation of a separated *PQI*–IO represents and how it is possible to obtain a "minimal" representation. With such results we define two algorithms, the first constructing a numerical representation for a given separated *PQI*–IO, the second minimising it. Both algorithms are shown to run in polynomial time ($\mathrm{O}(n^2)$ for the first and $\mathrm{O}(n)$ for the second).

Appendix A

Proof of Proposition 1. We provide the proofs for L (those of R are similar).

(i) $aQbLc \Rightarrow [(r(a) > r(b) \geqslant l(a) > l(b))$ and $(r(c) \geqslant r(b) \geqslant l(b) \geqslant l(c))] \Rightarrow r(c) \geqslant l(a) > l(c) \Rightarrow (a, c) \in Q \cup L$.

(ii) $aPbLc \Rightarrow [(l(a) > r(b))$ and $(r(c) \geqslant r(b) \geqslant l(b) \geqslant l(c))] \Rightarrow l(a) > l(c) \Rightarrow (a, c) \in P \cup Q \cup L$.

(iii) $aPbQ^{-1}c \Rightarrow [(l(a) > r(b))$ and $(r(c) > r(b) \geqslant l(c) > l(b))] \Rightarrow l(a) > l(c) \Rightarrow (a, c) \in P \cup Q \cup L$.

(iv) Otherwise, $\exists x$, $(x, x) \in (Q \cup L.Q \cup Q.L \cup L.Q.L).\hat{T}_l$. By Theorem 4.1 and (i), (ii) we have $(Q \cup L.Q \cup Q.L \cup L.Q.L) \subset (Q \cup P \cup L)$ and $(Q \cup P \cup L).P \subset P$.
We have $(x, x) \in (Q \cup L.Q \cup Q.L \cup L.Q.L).\hat{T}_l \subset (Q \cup P \cup L).P.\hat{I} \subset P.\hat{I} = \hat{T}_l$, impossible as $\hat{T}_l$ asymmetric.

(v) As $P = P.I_d \subset \hat{T}_l \subset T_l$ and $Q \subset T_l$ then $P \cup Q \subset T_l$.
$Q \cup L.Q \cup Q.L \cup L.Q.L \subset P \cup Q \cup L$ (Theorem 4.1 and (i)).
$\hat{T}_l = P.(I \cup Q \cup Q^{-1})$, $P.I \subset P$, $P.Q \subset P$ and $P.Q^{-1} \subset P \cup Q \cup L$ (by (iii)).
Therefore, $T_l \subset P \cup Q \cup L$.

(vi) Direct consequence of *v*.

(vii) $T_l.P \subset P$ and $P.T_r \subset P$
$T_l.P = P.\hat{I}.P \cup Q.P \cup L.Q.P \cup Q.L.P \cup L.Q.L.P \subset P$ (as $L.P \subset P$ and $Q.P \subset P$).

(viii) $P.T_l \subset T_l$ and $T_r.P \subset T_r$
$P.T_l = P.P.\hat{I} \cup P.Q \cup P.L.Q \cup P.Q.L \cup P.L.Q.L \subset P.\hat{I} \cup P \cup P.(P \cup \hat{I}) \subset T_l$. □

Proof of Theorem 4.3. We consider only T_l (T_r is similar).

(i) We show that T_l is asymmetric and negatively transitive.

Asymmetry: We recall that if R, S are two asymmetric relations and $R \cap S^{-1} = \emptyset$ then $R \cup S$ is asymmetric. P, Q, L are asymmetric and mutually exclusive $\Rightarrow (P \cup Q \cup L)$ is asymmetric $\Rightarrow Q_l \subset (P \cup Q \cup L)$ is asymmetric too. As $\hat{T}_l$ and Q_l are asymmetric, furthermore $Q_l \cap \hat{T}_l^{-1} = \emptyset$ (Proposition 1.(iv)) T_l is asymmetric.
Negative transitivity: We have to prove that $\neg T_l(a, b) \wedge \neg T_l(b, c) \wedge T_l(a, c)$ implies a contradiction. Since $T_l \subset P \cup Q \cup L$ and $\neg T_l \subset P^{-1} \cup Q^{-1} \cup L \cup L^{-1}$, we can eliminate the most trivial cases using this kind of verification $P^{-1}.P^{-1} \subset P^{-1} \notin \{P, Q, L\}, \ldots$. The other cases are considered in the following table.

(a,b)	(b,c)	(a,c)	Eliminated by
$P^{-1} \cup Q^{-1}$	L	L	$b(P \cup Q)aLc \Rightarrow (b,c) \in (P.L \cup Q.L) \subset (\hat{T}_l \cup Q_l) \subset T_l$
L	Q^{-1}	L	$aLcQb \Rightarrow (a,b) \in L.Q \subset Q_l \subset T_l$
L	R	$P \cup Q$	$a(P \cup Q)cLb \Rightarrow (a,b) \in (P.L \cup Q.L) \subset (\hat{T}_l \cup Q_l) \subset T_l$
R	L	Q	$bLaQc \Rightarrow (b,c) \in L.Q \subset Q_l \subset T_l$
L	L	L	non-trivial
L	R	L	non-trivial
R	L	L	non-trivial

The three last cases can be resumed by $(a, c) \in T_l \cap L \wedge (a, b) \in (L \cup R) \backslash T_l \wedge (b, c) \in (L \cup R) \backslash T_l$ with $(a, c) \in (T_l \cap L) = (\hat{T}_l \cup Q_l) \cap L = [P.(Id \cup Q \cup Q^{-1} \cup L \cup L^{-1}) \cup (Q \cup L.Q \cup Q.L \cup L.Q.L)] \cap L = (P.Q^{-1} \cap L) \cup (P.L \cap L) \cup (Q.L \cap L) \cup (L.Q.L \cap L)$.

- If $(a, c) \in (P.Q^{-1} \cup P.L) \cap L$ then $\exists x, aPx(Q^{-1} \cup L)c \wedge aLc$ i.e. $r(c) \geqslant r(a) \geqslant l(a) > r(x)$. If $l(b) > r(x)$ then $bPx(Q^{-1} \cup L)c \Rightarrow (b, c) \in T_l$. Otherwise, $l(b) \leqslant r(x)$ $\Rightarrow x(Q^{-1} \cup L)b$ (as $r(b) \geqslant l(a) > r(x)$). Then $aPx(Q^{-1} \cup L)b \Rightarrow (a, b) \in (P.Q^{-1} \cup P.L) \subset \hat{T}_l \subset T_l$.
- If $(a, c) \in Q.L \cap L$ then $\exists x, aQxLc \wedge aLc$ i.e. $r(c) \geqslant r(a) > r(x) > l(a) > l(x) > l(c)$. If $l(b) \geqslant l(a)$ then bLa (as $(a, b) \in (L \cup R)$) and $bLaQxLc \Rightarrow (b, c) \in T_l$. If $l(x) < l(b) < l(a)$ then $aLb \Rightarrow r(b) > r(a) > r(x) \Rightarrow bQx$ and $bQxLc \Rightarrow bT_lc$. Otherwise, $l(b) \leqslant l(x) < l(a) \Rightarrow aLb \Rightarrow r(b) > r(x) \Rightarrow xLb$. Then $aQxLb \Rightarrow (a, b) \in Q.L \subset Q_l \subset T_l$.
- If $(a, c) \in L.Q \cap L$ then $\exists x, aLxQc \wedge aLc$ i.e. $r(x) > r(c) \geqslant r(a) \geqslant l(a) \geqslant l(x) > l(c)$. If $l(b) \geqslant l(x)$ then $b(P \cup Q \cup L)xQc \Rightarrow bT_lc$. If $l(c) < l(b) < l(x)$ and $r(b) < r(x)$ then $aLxQb \Rightarrow (a, b) \in T_l$. If $l(c) < l(b) < l(x)$ and $r(b) \geqslant r(x)$ then bQc (as $r(x) > r(c)$) and bT_lc. Otherwise, $l(b) \leqslant l(c) \Rightarrow aL.Qc(P \cup Q \cup L)b \Rightarrow aT_lb$.
- If $(a, c) \in L.Q.L \cap L$ then $\exists x, y, \; aLxQyLc \wedge aLc$ i.e. $l(a) \geqslant l(x) > l(y) \geqslant l(c)$. If $l(b) \leqslant l(x) \Rightarrow b(P \cup Q \cup L)xQyLc \Rightarrow bT_lc$. If $l(y) < l(b) < l(x)$ and $r(b) < r(x)$ then $aLxQb \Rightarrow (a, b) \in T_l$. If $l(y) < l(b) < l(x)$ and $r(b) \geqslant r(x)$ then $r(b) > r(y)$ and $bQyLc \Rightarrow (b, c) \in T_l$. Otherwise, if $l(b) \leqslant l(y)$ then $aLxQy(P \cup Q \cup L)b \Rightarrow (a, b) \in T_l$.

(ii) Immediate from Theorems 2.1, 2.2 and (i).

(iii) $T_l^{\sim} \cap T_r^{\sim} \subset E$. If $(x, y) \in T_l^{\sim} \cap T_r^{\sim} \Rightarrow (x, y) \notin T_l \cup T_l^{-1} \cup T_r \cup T_r^{-1}$. Suppose that $(x, y) \notin E$ then $\exists z \in A, \; zR_1x$ and zR_2y with $R_1 \neq R_2$. Consider, for example,

$R_1 = P$, we have:
$zP^{-1}y \Rightarrow yPzPx \Rightarrow yT_lx$, impossible.
$zQy \Rightarrow yQ^{-1}zPx \Rightarrow y\hat{I}.Px \Rightarrow yT_rx$, impossible.
$zQ^{-1}y \Rightarrow yQzPx \Rightarrow yT_lx$, impossible.
The other cases are quite similar.

(iv) Immediate from $\hat{T}_l \subset T_l$ and $\hat{T}_r \subset T_r$. $\square$

Proof of Theorem 4.4.

(i) We first demonstrate that T_0 is asymmetric and negatively transitive.
Asymmetry: $T_0 = (T_0 \cap A_l \times A_l) \cup (T_0 \cap A_r \times A_r) \cup (\hat{T}_0 \cap (A_l \times A_r \cup A_r \times A_l))$, where $(T_0 \cap A_l \times A_l)$ (resp. $(T_0 \cap A_r \times A_r)$) is in fact isomorph to T_l (resp. T_r). As each component of T_0 is asymmetric and belongs to, respectively, $A_l \times A_l$, $A_r \times A_r$, $A_l \times A_r \cup A_r \times A_l$ which are mutually exclusive, T_0 is asymmetric.
Negative transitivity: $\neg T_0(x, y), \neg T_0(y, z)$. x, y, z can be a_l or a_r, b_l or b_r, c_l or c_l respectively. There exist eight possible combinations, but four of them are the inverse of the other four. Thus, we only have to prove these four cases.
Case 1: $a_l \neg T_0 b_l \neg T_0 c_l \Rightarrow a_l \neg T_l b_l \neg T_l c_l$ (by definition).
$\Rightarrow a_l \neg T_l c_l$, ($T_l$ is a weak order).
$\Rightarrow a_l \neg T_0 c_l$, (by definition).
Case 2: $a_l \neg T_0 b_l \neg T_0 c_r \Rightarrow a_l \neg T_0 c_r$ i.e. $a \neg T_l b, \neg P(b, c) \Rightarrow \neg P(a, c)$
i.e. $P(a, c), \neg P(b, c) \Rightarrow T_l(a, b)$ where $\neg P = P^{-1} \cup Q \cup Q^{-1} \cup I = P^{-1} \cup \hat{I}$
$P(a, c), \neg P(b, c) \Rightarrow (a, b) \in P.(P \cup \hat{I}) \subset T_l$
Case 3: $a_l \neg T_0 b \neg T_0 c_l \Rightarrow a_l \neg T_0 c_l$ i.e. $\neg P(a, b), P(c, b) \Rightarrow \neg T_l(a, c)$
i.e. $T_l(a, c), P(c, b) \Rightarrow P(a, b)$
$T_l(a, c), P(c, b) \Rightarrow (a, b) \in T_l.P \subset P$, (by Proposition 1.vii).
Case 4: $a_l \neg T_0 b_r \neg T_0 c_r \Rightarrow a_l \neg T_0 c_r$ i.e. $\neg P(a, b), \neg T_r(b, c) \Rightarrow \neg P(a, c)$
Similar to case 2.

(ii) Immediate from Theorems 2.1, 2.2 and (i).

(iii) Consider $[x]_{T_0^{\sim}}$, $x \in A_l \cup A_r$. We will demonstrate that "if $x = a_l (x = a_r)$ for some $a \in A$ then $[x]_{T_0^{\sim}} = [a_l]_{T_l^{\sim}}$ ($[x]_{T_0^{\sim}} = [a_r]_{T_r^{\sim}}$)". By construction of T_0, if $\neg T_0(x, y)$ and $\neg T(y, x)$ then $(x, y) \notin A_l \times A_r \cup A_r \times A_l$. Suppose that $x = a_l$, if $y \in [x]_{T_0^{\sim}}$ then $y = b_l$ for some $b \in A$, and $\neg T_0(a_l, b_l)$ and $\neg T_0(b_l, a_l) \Leftrightarrow \neg T_l(a_l, b_l)$ and $\neg T_l(b_l, a_l)$ $\Leftrightarrow b_l \in [a_l]_{T_l^{\sim}}$. The case $x = a_r$ is similar. $\square$

Proof of Proposition 2.

(i) $T_0(a_l, b_l) \Rightarrow T_l(a, b) \Rightarrow (a, b) \in P \cup P.Q \cup P.Q^{-1} \cup P.L \cup P.R \cup Q \cup L.Q \cup Q.L \cup L.Q.L \subset P \cup Q \cup P.L \cup L.Q \cup Q.L \cup L.Q.L$. If aPb then $l(a) \geqslant r(b) + \varepsilon \geqslant l(b) + \varepsilon$.
If aQb then $l(a) \geqslant l(b) + \varepsilon$.
If $aPcLb$ then $l(a) \geqslant r(c) + \varepsilon \geqslant l(c) + \varepsilon \geqslant l(b) + \varepsilon$.
If $aLcQb$ then $l(a) \geqslant l(c) \geqslant l(b) + \varepsilon$.
If $aQcLb$ then $l(a) \geqslant l(c) + \varepsilon \geqslant l(b) + \varepsilon$.
If $aLcQdLb$ then $l(a) \geqslant l(c) \geqslant l(d) + \varepsilon \geqslant l(b) + \varepsilon$.

(ii) Similar to (i).
(iii) $T_0(a_l, b_r) \Leftrightarrow P(a, b) \Rightarrow l(a) \geqslant r(b) + \varepsilon$.
(iv) $T_0(a_r, b_l) \Leftrightarrow \neg P(b, a) \Rightarrow r(a) \geqslant l(b)$. $\square$

Proof of Theorem 4.6. We consider the valued graph $G = ((A_l \cup A_r)/T_0^{\sim}, T_0, v)$ where:

$$v(x, y) = \begin{cases} 0 & \text{if } x = [a_r],\ y = [b_l] \text{for some } a, b \in A, \\ \varepsilon & \text{otherwise.} \end{cases}$$

T_0 is a linear order $\Rightarrow$ there is no circuit $\Rightarrow$ a potential function exists (Theorem 2.3). We prove that the maximal value of the paths starting from a node a_l (a_r) (being also the smallest potential function) is: $g(a_l) = l^*(a)$, $g(a_r) = r^*(a)$. The nodes of G can be presented as $Z_{l+r}T_0Z_{l+r-1}T_0 \ldots Z_1$. Remind that $Z_iT_0Z_j$ iff $i \geqslant j$ and all the arcs of G are either 0 or $\varepsilon > 0$. By Proposition 2 and Theorem 4.5, in two consecutive arcs, there is at least one arc with value ε. For each Z_k, consider the path $\Phi = Z_kT_0 \ldots Z_1$ and denote $V(\Phi)$ its value. Any other path Φ' starting from Z_k is obtained from Φ by applying (recursively) the following operation:

- drop out the last arc (x, y), obviously $V(\Phi) \geqslant V(\Phi')$ ($v(x, y) \geqslant 0$).
- replace a portion $(Z_i, Z_{i-1}, \ldots Z_j)$ by (Z_i, Z_j). As $V(Z_i, Z_j) \leqslant \varepsilon$ and $V(Z_i, Z_{i-1}, \ldots Z_j) \geqslant \varepsilon$ then $V(\Phi) \geqslant V(\Phi')$. Thus, Φ is the path with maximal value starting from Z_k.

By Theorem 4.5, along Φ, all the arcs are ε, but (a_r, b_l) which are transitive arcs connecting B_j to A_j. If $Z_i = a_l \in A_j$, there exist $(j-1)$ transitive arcs $\Rightarrow V(\Phi) = (i - j + 1) * \varepsilon$. If $Z_i = a_r \in B_j$, there exist j transitive arcs $\Rightarrow V(\Phi) = (i - j) * \varepsilon$. $\square$

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